Nonexistence of colimits in naive discrete homotopy theory

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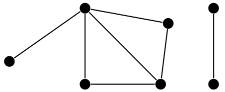
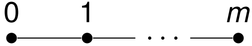
First definitions

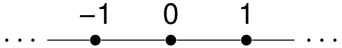
- Graph = set w/ symmetric reflexive relation $\sim$ (denoted $G, H, \dots$)



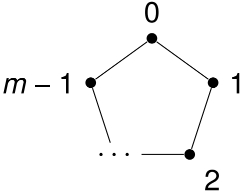
I_m



I_∞



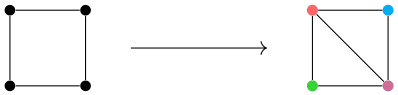
C_m

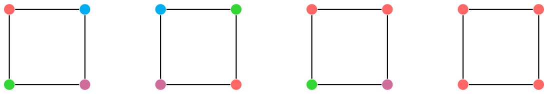


First definitions

- Graph map (denoted $G \to H$) = function preserving $\sim$
  - reflexivity $\implies$ graph maps can collapse edges

e.g.



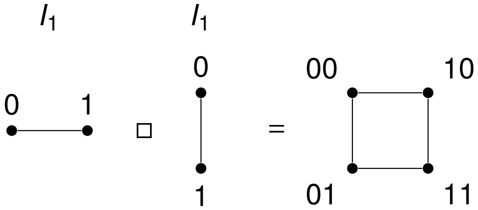


Box product

Definition

The *box product* $G \square H$ has

- vert: pairs $(v \in G, w \in H)$
- edge: $(v, w) \sim (v', w')$ if $\begin{matrix} v \sim v' \\ w = w' \end{matrix}$ or $\begin{matrix} v = v' \\ w \sim w' \end{matrix}$



Homotopy

Definition

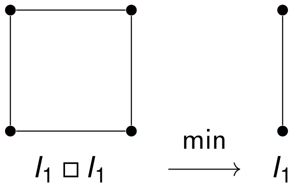
- Given $f, g\colon G \to H$, a *homotopy* is $\alpha\colon G \square I_m \to H$ for some $m \geq 0$ s.t.

$$\alpha(-, 0) = f \text{ and } \alpha(-, m) = g$$

- A map $f\colon G \to H$ is a *homotopy equivalence* if there exist:
  - $g\colon H \to G$
  - homotopy $gf \sim \mathrm{id}_G$
  - homotopy $fg \sim \mathrm{id}_H$

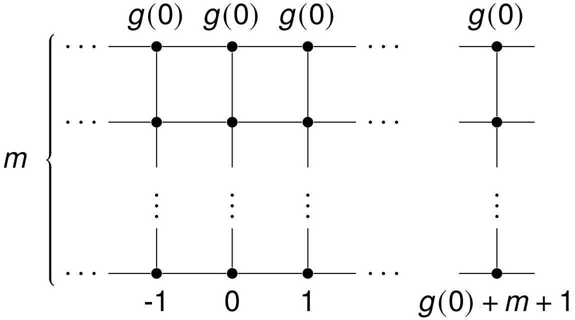
Examples

- $f\colon I_m \to I_0$ is htpy equiv
  - $g\colon I_0 \to I_m$ picks 0
  - $gf \sim \mathrm{id}_{I_0}$ easy
  - $fg \sim \mathrm{id}_{I_m}$ is $\min\colon I_m \,\square\, I_m \to I_m$



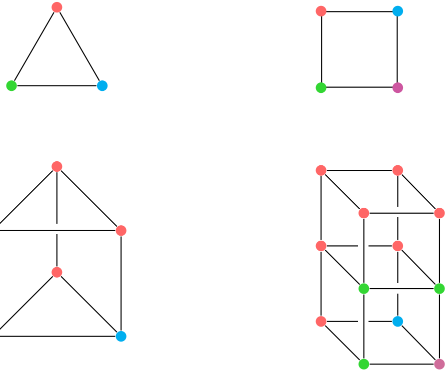
Examples

- $f: I_\infty \longrightarrow I_0$ is *not* htpy equiv
  - $g: I_0 \longrightarrow I_\infty$
  - $fg \nsim \mathrm{id}_{I_\infty}$



Examples

- $\begin{array}{l} C_3 \to I_0 \\ C_4 \to I_0 \end{array}$ are htpy equivs



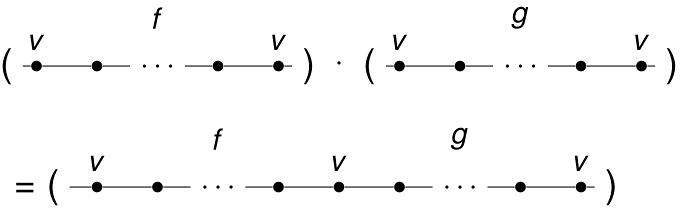
Discrete fundamental group

- $C_5 \to I_0$ is *not* htpy equiv

Definition

The *fundamental group* of (G, v) is

$$A_1(G,v) := \{f\colon I_\infty \to G \mid f(i) = v \text{ for almost all } i \in I_\infty\} \Big/ \sim_*$$



Discrete fundamental group

Proposition

The fundamental group is a homotopy invariant □

Remark

If v, w path-connected then $A_1(G, v) \cong A_1(G, w)$.

Proposition

$$A_1(C_M) \cong \begin{cases} \{*\} & M = 3, 4 \\ \mathbb{Z} & M \geq 5. \end{cases}$$

Proof.

1. Develop covering spaces
2. Show $I_\infty \to C_M$ is universal cover (for $M \geq 5$) □

Homotopy groups

Definition

For $n \geq 0$, the *n-th homotopy group* of (G, v) is

$$A_n(G, v) := \{f\colon I_\infty^{\square n} \to G \mid f(i) = v \text{ for almost all } i \in I_\infty^{\square n}\} \Big/ \sim_*$$

Proposition

For $G = I_m, I_\infty, C_m$,

$$A_n(G) \cong \{*\}$$

for any $n \geq 2$ and $v \in G$. □

How do we get non-trivial higher homotopy groups?

Homotopy groups

Definition

The *length-m suspension* of G is

$$S_mG := G \,\square\, I_m \Bigg/ \begin{array}{l}(v,0)\sim(v',0)\\(v,m)\sim(v',m)\end{array}$$

Theorem (C.–Kapulkin, Barcelo–Greene–Jarrah–Welker)

Suppose

- *G is connected*
- *$n \geq 1$ is minimal s.th. $A_n(G)$ non-trivial*

then there exists $m \geq 1$ s.th.

1. $A_n(S_mG) \cong \{*\}$
2. $A_{n+1}(S_mG) \cong A_n(G)$ □

Homotopy groups

Corollary (Lutz)

For any $n \geq 0$, there exists G s.th. $A_n(G) \neq \{*\}$

Proof.

Apply previous theorem to C_5 □

Remark

Can upgrade this (due to Kapulkin–Mavinkurve)

1. For any group Γ, there exists G s.th. $A_1(G, v) \cong \Gamma$
2. For any abelian group Γ and $n \geq 2$, there exists G s.th.
   - $A_n(G, v) \cong \Gamma$
   - $A_k(G, v) \cong \{*\}$ for $k < n$

How to import more theorems?

- Question (AIM Workshop): are htpy equivs the weak equivalences in some model structure on graphs?
- Motivation: understand homotopy pushouts

From topology

For a homotopy pushout

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & P \end{array} \quad (\ulcorner_h)$$

- (van-Kampen) $\Pi_1(P)$ is a (strict) pushout ...
- (Mayer-Vietoris) there is a LES in homology ...

How to import more theorems?

- Currently have partial analogues and/or ad hoc conditions
  - harder to import new theorems (e.g. excision, Blakers–Massey, homology is determined by axioms, …)
  - harder to compute (e.g. homology of S_5C_5, $\partial I_5^{\square 3}$, … ?)

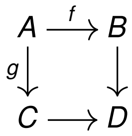
- Understand homotopy pushouts $\rightsquigarrow$ solve both problems

- Model categories come w/ tools for homotopy pushouts

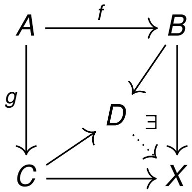
Homotopy pushouts

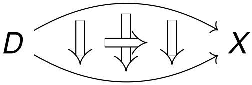
Definition (informally)

A (htpy-commutative) square



is a *homotopy pushout* if





. . .

Homotopy pushouts

- Formal definition is a universal property in some ∞-cat
  - Fixed by choice of weak equivalences
- Examples in ∞-category of spaces:

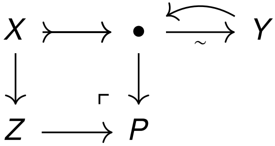
$$\begin{array}{ccc} X & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & SX \end{array} \;(\ulcorner_h) \qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow \\ Z & \longrightarrow & C(f,g) \end{array} \;(\ulcorner_h)$$

- Model category has three classes of maps
  - weak equivalences (fix homotopy pushout & pullback)
  - cofibrations (how to compute homotopy pushouts)
  - fibrations (how to compute homotopy pullbacks)

  subject to axioms

Homotopy pushouts

In any model category,

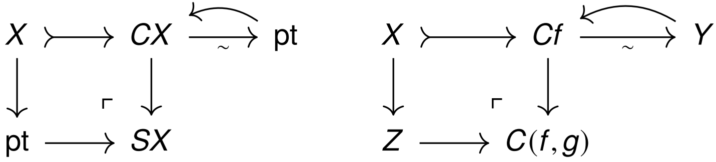


Theorem (C.–Kapulkin–Kim)

The homotopy theory (*Graphs*, *htpy equiv's*) *does not have homotopy pushouts.*
In particular, there is no model structure on graphs whose weak equivalences are htpy equiv's

Homotopy pushouts

In any model category (e.g. topological spaces),

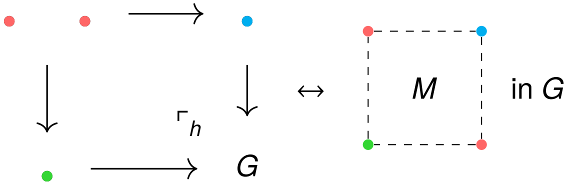


Theorem (C.–Kapulkin–Kim)

The homotopy theory $(\mathit{Graphs}, \mathit{htpy\ equiv's})$ *does not have homotopy pushouts.*
In particular, there is no model structure on graphs whose weak equivalences are htpy equiv's

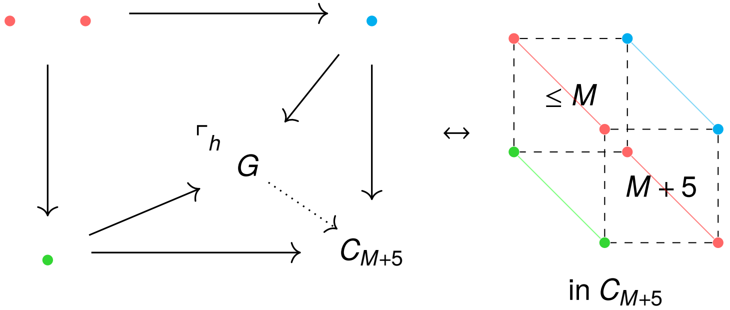
Proof of Thm

Goal: missing a homotopy pushout



- Suppose G is homotopy pushout
- M := length of cycle in G

Proof of Thm



- Homotopy on left cannot exist in C_{M+5}
  – contradiction! □

Moral of the story

- Look at weak equiv's, not htpy equiv's

Definition
A map $f\colon G \to H$ is a *weak equivalence* if

$$A_n f\colon A_n(G, v) \to A_n(H, fv)$$

is iso for all $n \geq 0$ and $v \in G$.

Conjecture (C.–Kapulkin)
The homotopy theory $(\text{Graphs}, \text{weak equiv's})$ has all homotopy pushouts

Key: $S_{m+1}G \to S_m G$ is weak equiv

Thank you!