Cubical setting for discrete homotopy theory

Daniel Carranza
University of Toronto

Chris Kapulkin
University of Western Ontario

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Discrete homotopy theory

- Discrete homotopy theory is a homotopy theory for graphs
- Graphs are simple, undirected, no self-edges

Definition (from topology)

For $f, g \colon X \to Y$, a *homotopy* from f to g is $\alpha \colon X \times [0,1] \to Y$ s.t.

$$\alpha(x,0) = f(x) \text{ and } \alpha(x,1) = g(x)$$

for all $x \in X$.

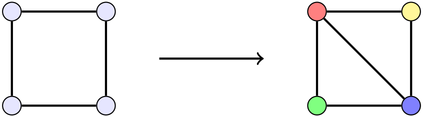
- We need notions of:
  1. graph maps
  2. product
  3. interval

Graph maps

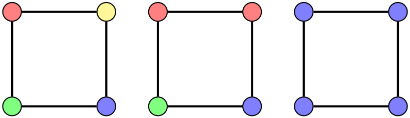
A *graph map* $f\colon G \to H$ is $f\colon G_V \to H_V$ s.th.

$$v \sim w \implies f(v) \sim f(w) \text{ or } f(v) = f(w).$$

Example



e.g.

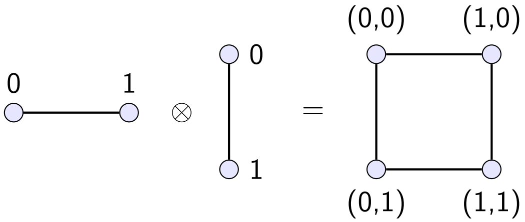


Cartesian Product

The *Cartesian product* $G \otimes H$ has

- vertices: pairs $(g \in G, h \in H)$
- $(v, w) \sim (v', w')$ if $v = v'$ and $w \sim w'$
  or $v \sim v'$ and $w = w'$

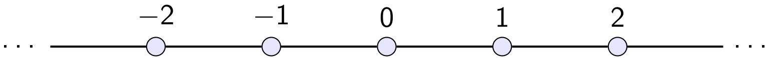
Example



Interval(s)

I_m

I_∞



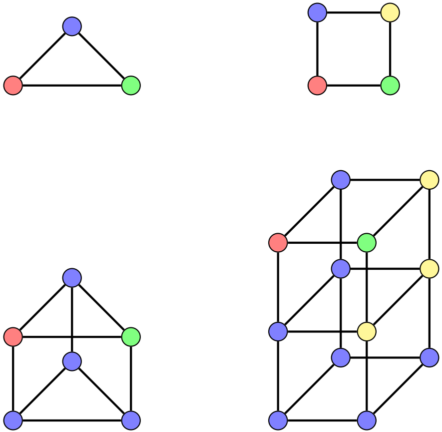
Discrete homotopies

- A *homotopy* from f to g is $\alpha\colon G \otimes I_m \to H$ (for some $m \geq 0$) s.t.
$$\alpha(v, 0) = f(v) \text{ and } \alpha(v, m) = g(v)$$
for all $v \in G$.

- A *homotopy equivalence* is $f\colon G \to H$ with $g\colon H \to G$ and homotopies
$$\mathrm{id}_G \sim gf \quad \text{and} \quad \mathrm{id}_H \sim fg.$$

Not just Hatcher

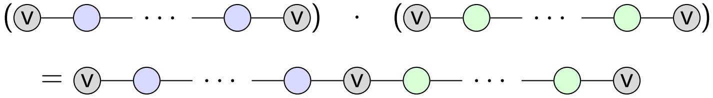
$C_3 \to I_0$ and $C_4 \to I_0$ are homotopy equivalences.



Discrete homotopy groups

The *n-th homotopy group* of (G, v) is

$$A_n(G, v) := \{f : I_\infty^{\otimes n} \to G \mid f(\vec{n}) = v \text{ for almost all } \vec{n} \in I_\infty^{\otimes n}\} / \sim_* .$$



Theorem

$$A_1(C_m, 0) \cong \begin{cases} 1 & m = 3, 4 \\ \mathbb{Z} & m \geq 5. \end{cases}$$

□

Applications

- Social and technological networks ~1974

  Atkin, *An algebra for patterns on a complex*

- Hyperplane arrangements ~2005

  Babson, Barcelo, Laubenbacher, *A-homotopy theory and arrangements of linear subspaces*

- Matroids 2015

  Chalopin, Chepoi, Osajda, *On two conjectures of Maurer concerning basis graphs of matroids*

- Topological data analysis 2019

  Mémoli, Zhou, *Persistent Homotopy Groups of Metric Spaces*

- Geometric group theory ~2020

  Delabie, Khukhro, *Course fundamental groups and box spaces*

Cubical setting

For a graph G, construct a CW-complex X_G:

- 0-cells: vertices of G i.e. maps $I_0 \to G$
- 1-cells: maps $I_1 \to G$
- 2-cells: maps $I_1^{\otimes 2} \to G$
- 3-cells: maps $I_1^{\otimes 3} \to G$

  ⋮
- n-cells: maps $I_1^{\otimes n} \to G$

  ⋮

Conjecture (Babson, Barcelo, de Longueville, Laubenbacher)

$$A_n(G, v) \cong \pi_n(X_G, v).$$

Cubical Setting

- Use cubical sets, not spaces

Theorem (C, Kapulkin)

- $A_n(G, v) \cong \pi_n(X_G, v)$
- *Hurewicz theorem*
- *Long exact sequence of A-homotopy groups*

Thank you!